

LEFT ORDERS IN GARSIDE GROUPS

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ABSTRACT. We consider the structure group of a non-degenerate symmetric (non-trivial) set-theoretical solution of the quantum Yang-Baxter equation. This is a Bieberbach group and also a Garside group. We show this group is not bi-orderable, that is it does not admit a total order which is invariant under left and right multiplication. Regarding the existence of a left invariant total ordering, there is a great diversity. There exist structure groups with a recurrent left order and with space of left orders homeomorphic to the Cantor set, while there exist others that are even not unique product groups.

INTRODUCTION

A group G is *left-orderable* if there exists a strict total ordering $<$ of its elements which is invariant under left multiplication, that is $g < h$ implies $fg < fh$ for all f, g, h in G . If the order $<$ is also invariant under right multiplication, then G is said to be *bi-orderable*. The braid group B_n , with $n \geq 3$ strands, is left-orderable but not bi-orderable [16], and if $n \geq 5$ none of these orders is Conradian [34]. In [17], the question whether every Garside group is left-orderable is raised (Question 3.3, p.292, also in [18]). It is a very natural question as the Garside groups extend the braid groups in many respects and it motivated our research in the context of the structure group of a non-degenerate symmetric set-theoretical solution of the quantum Yang-Baxter equation. This group is a Garside group that satisfies many interesting properties [6], [15]. In this note, we show this group is not bi-orderable and we find the question whether it is left-orderable has a wide range of answers. We state our main results and refer to Sections 1,2 for definitions:

Theorem 1. *Let $G(X, S)$ be the structure group of a non-degenerate symmetric (non-trivial) set-theoretical solution (X, S) of the quantum Yang-Baxter equation. Then $G(X, S)$ is not bi-orderable. Furthermore, $G(X, S)$ has generalized torsion elements.*

Theorem 2. *Let $G(X, S)$ be the structure group of a non-degenerate, symmetric (non-trivial) set-theoretical solution (X, S) of the quantum Yang-Baxter equation. Assume (X, S) is a retractable solution and $|X| \geq 3$. Then*

- (i) $G(X, S)$ has a recurrent left order.
- (ii) The space of left orders of $G(X, S)$ is homeomorphic to the Cantor set.
- (iii) $G(X, S)$ has an infinite number of Conradian left orders.

Note that under the assumptions of Theorem 2, $G(X, S)$ is locally indicable (each non-trivial finitely generated subgroup has a quotient isomorphic to \mathbb{Z}), as the existence of a recurrent left order implies local indicability [32]. In contrast, for $n \geq 5$, no braid group B_n is locally indicable [17][p.287] and hence B_n has no recurrent left order like most of the left-orderable groups. E. Jespers and J. Okninski prove the structure group of a retractable solution is poly-(infinite)cyclic [24][p.223] and poly-(infinite)cyclic implies locally indicable. Here is an outline of the paper. Section 1 provides some standard definitions on orderable groups. Section 2 introduces the structure group of a non-degenerate, symmetric set-theoretical solution (X, S) of the quantum Yang-Baxter equation. Section 3 proves Theorem 1. Section 4 proves Theorem 2 and concludes the paper with some remarks and questions.

1. PRELIMINARIES ON GROUPS ORDERING

We introduce some definitions and refer to [26], [22], [17], [1], [18], [28], [29] and survey [36]. A group G is *left-orderable* if there exists a strict total ordering \prec of its elements which is invariant under left multiplication, that is $g \prec h$ implies $fg \prec fh$ for all f, g, h in G . If a group G is left-orderable, then it satisfies the unique product property, that is for any finite subsets $A, B \subseteq G$, there exists at least one element $x \in AB$ that can be uniquely written as $x = ab$, with $a \in A$ and $b \in B$. We call a strict total ordering which is invariant under left multiplication a *left order*. The *positive cone* of a left order \prec is defined by $P = \{g \in G \mid 1 \prec g\}$ and it satisfies:

- (1) P is a semigroup, that is $P \cdot P \subseteq P$
- (2) G is partitioned by P , that is $G = P \cup P^{-1} \cup \{1\}$ and $P \cap P^{-1} = \emptyset$

Conversely, if there exists a subset P of G that satisfies (1) and (2), then P determines a unique left order \prec defined by $g \prec h$ if and only if $g^{-1}h \in P$. A subgroup N of a left-orderable group G is called *convex* (with respect to \prec), if for any $x, y, z \in G$ such that $x, z \in N$ and $x \prec y \prec z$, we have $y \in N$. A left order \prec is *Conradian* if for any strictly positive elements $a, b \in G$, there is $n \in \mathbb{N}$ such that $b \prec ab^n$. A left-orderable group G is called *Conradian* if it admits a Conradian left order. Conradian left-orderable groups share many of the properties of the bi-orderable groups. A left order \prec in a countable group G is *recurrent* (for every cyclic subgroup) if for every $g \in G$ and every finite increasing sequence $h_1 \prec h_2 \prec \dots \prec h_r$ with $h_i \in G$, there exists $n_i \rightarrow \infty$ such that $\forall i, h_1 g^{n_i} \prec h_2 g^{n_i} \prec \dots \prec h_r g^{n_i}$ (see [32][Defn.3.2], [31][Defn.3.1]). A recurrent left order is Conradian [32].

The set of all left orders of a group G is denoted by $LO(G)$ and it is a topological space (compact and totally disconnected with respect to the topology induced by the Tychonoff topology on the power set of G) [37]. If G is left-orderable, it acts on $LO(G)$ by conjugation: the image of \prec under $g \in G$ is $\prec_g \in LO(G)$ defined by $a \prec_g b$ if and only if $gag^{-1} \prec gbg^{-1}$. A left order \prec is *finitely determined* if there is a finite subset $\{g_1, g_2, \dots, g_k\}$ of G such that \prec is the unique left-invariant ordering of G satisfying $1 \prec g_i$ for $1 \leq i \leq k$. A finitely determined left order \prec is also called *isolated*, since \prec is finitely determined if and only if it is not a limit point of $LO(G)$. If the positive cone of \prec is a finitely generated semigroup, then \prec is isolated. The set $LO(G)$ cannot be countably infinite [29]. If G is a countable left-orderable group, $LO(G)$ is either finite, or homeomorphic to the Cantor set, or homeomorphic to a subspace of the Cantor space with isolated points. Furthermore, $LO(G)$ is homeomorphic to the Cantor set if and only if it is nonempty and no left-invariant ordering of G is isolated [17][p.267]. If G is a countable and virtually solvable left-orderable group, then $LO(G)$ is either finite or a Cantor set [35]. An element x of a subset F of G is an *extreme point* of F if, for all $g \in G \setminus \{1\}$, either ga or $g^{-1}a$ is not in F . A group G is *diffuse* if every non-empty finite subset F of G has an extreme point [2], [31]. A diffuse group satisfies the unique product property [2]. This notion is strictly weaker than left-orderability [25] and it is equivalent to the notion of locally invariant left order [31]. We refer to [31][Section 6].

2. SET-THEORETICAL SOLUTIONS OF THE YANG-BAXTER EQUATION

Fix a finite dimensional vector space V over the field \mathbb{R} . The quantum Yang-Baxter equation (QYBE) on V is the equality $R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}$ of linear transformations on $V \otimes V \otimes V$, where $R : V \otimes V \rightarrow V \otimes V$ is a linear operator and R^{ij} means R acting on the i th and j th components. A *set-theoretical solution* of this equation is a pair (X, S) such that X is a basis for V and $S : X \times X \rightarrow X \times X$ is a bijective map. The map S is defined by $S(x, y) = (g_x(y), f_y(x))$, where $f_x, g_x : X \rightarrow X$ are functions for all $x, y \in X$. The pair (X, S) is *non-degenerate* if for any $x \in X$, f_x and g_x are bijective. It is *involution* if $S \circ S = Id_X$, and *braided* if $S^{12}S^{23}S^{12} = S^{23}S^{12}S^{23}$, where the map S^{ii+1} means S acting on the i -th and $(i+1)$ -th components of X^3 . It is said to be *symmetric* if it is involutive

and braided. Let $\alpha : X \times X \rightarrow X \times X$ be defined by $\alpha(x, y) = (y, x)$, and let $R = \alpha \circ S$, then R satisfies the QYBE if and only if (X, S) is braided. We follow [19], [24] and refer to [20, 23] for more details, and to [6, 8, 9] for examples.

Definition 2.1. Let (X, S) be a non-degenerate symmetric set-theoretical solution. The *structure group* of (X, S) is $G(X, S) = \text{Gp}\langle X \mid xy = g_x(y)f_y(x) ; x, y \in X \rangle$.

Definition 2.2. [19] Let (X, S) be a non-degenerate symmetric solution.

- (i) A subset Y of X is an *invariant* subset if $S(Y \times Y) \subseteq Y \times Y$.
- (ii) An invariant subset Y is *non-degenerate* if $(Y, S|_{Y \times Y})$ is non-degenerate and symmetric.
- (iii) The solution (X, S) is *decomposable* if X is the union of two non-empty disjoint non-degenerate invariant subsets. Otherwise, (X, S) is *indecomposable*.

A solution (X, S) is *trivial* if $g_x = f_x = \text{Id}_X$, $\forall x \in X$, and its structure group is $\mathbb{Z}^{|X|}$. Non-degenerate symmetric solutions (up to equivalence) are in one-to-one correspondence with quadruples (G, X, ρ, π) , where G is a group, X is a set, ρ is a left action of G on X , and π is a bijective 1-cocycle of G with coefficients in \mathbb{Z}^X , where \mathbb{Z}^X is the free abelian group spanned by X [19]. Indeed, $G(X, S)$ is naturally a subgroup of $\text{Sym}(X) \ltimes \mathbb{Z}^X$, such that the 1-cocycle defined by the projection $G(X, S) \rightarrow \mathbb{Z}^X$ is bijective. The product in $\text{Sym}(X) \ltimes \mathbb{Z}^X$ is defined by: $f_x^{-1}t_x f_y^{-1}t_y = f_x^{-1}f_y^{-1}t_{f_y(x)}t_y$. More precisely:

Theorem 2.3. [19] Let (X, S) be a non-degenerate symmetric solution and $G(X, S)$ be its structure group. Let $\text{Sym}(X)$ be the group of permutations of X and \mathbb{Z}^X be the free abelian group spanned by X . Let the map $\phi_f : G(X, S) \rightarrow \text{Sym}(X) \ltimes \mathbb{Z}^X$ be defined by $\phi_f(x) = f_x^{-1}t_x$, where $x \in X$ and t_x is the generator of \mathbb{Z}^X corresponding to x . Then

- (i) The assignment $x \mapsto f_x^{-1}$ is a left action of $G(X, S)$ on X .
- (ii) Let $a \in G(X, S)$ and $w = m_1t_1 + m_2t_2 + \dots + m_nt_n \in \mathbb{Z}^X$. Assume a acts on X via the permutation f . Then a acts on \mathbb{Z}^X in the following way: $a \bullet t_x = t_{f(x)}$ and $a \bullet w = m_1t_{f(1)} + m_2t_{f(2)} + \dots + m_nt_{f(n)}$, where \bullet denotes the extension of the left action of $G(X, S)$ on X defined in (i) to \mathbb{Z}^X .
- (iii) The map ϕ_f is a monomorphism.
- (iv) The map $\pi : G(X, S) \rightarrow \mathbb{Z}^X$, defined by $\pi(g) = w$ if $\phi_f(g) = \alpha w$, with $\alpha \in \text{Sym}(X)$ and $w \in \mathbb{Z}^X$, is a bijective 1-cocycle satisfying $\pi(a_1a_2) = (a_2^{-1} \bullet \pi(a_1)) + \pi(a_2)$.

A *crystallographic group* is a discrete cocompact subgroup of the group of isometries of \mathbb{R}^n . A *Bieberbach group* is a torsion-free crystallographic group. The structure group $G(X, S)$ of a non-degenerate symmetric solution (X, S) with $|X| = n$ is a Bieberbach group of rank n [20]. Indeed, $G(X, S)$ acts freely on \mathbb{R}^n by isometries with fundamental domain $[0, 1]^n$ (see [20], [24][p.218]). The structure groups satisfy another property, that makes this family of groups particularly interesting. Indeed, every structure group is a *Garside group* [6], [15], that is a group of fractions of a cancellative monoid M which is a lattice with respect to left-divisibility and with a Garside element Δ (the left and right generators of Δ coincide, are finite in number and generate M). Garside groups were defined as a generalisation in some extent of the braid groups and the finite-type Artin-Tits groups [13], [15].

3. THE STRUCTURE GROUP IS NOT BI-ORDERABLE

If (X, S) is a non-degenerate symmetric set-theoretical solution, with $|X| = n$, that satisfies a certain condition C , then there is a short exact sequence $1 \rightarrow N \rightarrow G(X, S) \rightarrow W \rightarrow 1$, where N is a normal free abelian subgroup of rank n and W is a finite group of order 2^n . Moreover, W is a Coxeter-like group, that is W is a finite quotient that plays the role the pure braid group P_n plays in the sequence $1 \rightarrow P_n \rightarrow B_n \rightarrow S_n \rightarrow 1$, where B_n is the braid group and S_n the symmetric group or more generally the role Coxeter groups play for the finite-type Artin groups [8]. In [14], it is proved that the condition C may be relaxed and

that for each (X, S) there is a natural number m such that for each $x \in X$ there is a chain of trivial relations of the form $xy_1 = xy_1$, $y_1y_2 = y_1y_2$, $y_2y_3 = y_2y_3, \dots, y_{m-1}x = y_{m-1}x$, $y_i \in X$. Here, we call the element $y_1y_2y_3\dots y_{m-1}x$ the *frozen element of length m ending with x* and denote it by θ_x . The subgroup N , generated by the n frozen elements of length m , $y_1y_2y_3\dots y_{m-1}x$, $xy_1y_2y_3\dots y_{m-1}$, $\dots, y_2y_3\dots y_{m-1}xy_1$, is normal, free abelian of rank n and the group W defined by $G(X, S)/N$ is finite of order m^n and is a Coxeter-like group [14].

Lemma 3.1. *Let (X, S) be a non-degenerate symmetric set-theoretical solution, with $|X| = n$ and with permutations $f_1, \dots, f_n, g_1, \dots, g_n$. Let $G(X, S)$ be its structure group. Let $\theta_1, \dots, \theta_n$ be the n frozen elements in $G(X, S)$ and x_k be an element of X . Then*

(i) $x_k\theta_i x_k^{-1} = \theta_{f_k^{-1}(i)}$.

(ii) *If (X, S) is not the trivial solution, then there exists $1 \leq i \leq n$, such that $C_{G(X, S)}(\theta_i)$, the centralizer of θ_i , is not $G(X, S)$.*

Proof. (i) We show $\pi(x_k\theta_i) = \pi(\theta_{f_k^{-1}(i)}x_k)$, where $\pi : G(X, S) \rightarrow \mathbb{Z}^X$ is the bijective 1-cocycle defined in Theorem 2.3. Indeed, $\pi(x_k\theta_i) = \theta_i^{-1} \bullet \pi(x_k) + \pi(\theta_i) = \pi(x_k) + \pi(\theta_i) = t_k + mt_i$, using first Theorem 2.3(iv), next the fact that the frozen elements act trivially on \mathbb{Z}^n and that $\pi(\theta_i) = mt_i$ (see [14] lemma 3.2 or [9] lemma 1.12 written multiplicatively there). In the same way, $\pi(\theta_{f_k^{-1}(i)}x_k) = x_k^{-1} \bullet \pi(\theta_{f_k^{-1}(i)}) + \pi(x_k) = x_k^{-1} \bullet (mt_{f_k^{-1}(i)}) + t_k = mt_{f_k^{-1}(i)} + t_k = mt_i + t_k$.

(ii) From (i), $C_{G(X, S)}(\theta_i) = G(X, S)$ for all $1 \leq i \leq n$, if and only if $f_k^{-1}(i) = i$ for all $1 \leq i, k \leq n$, that is if and only if (X, S) is the trivial solution. \square

An element $g \neq 1$ of a group G is called a *generalized torsion element* if there exist $h_i \in G$ such that $\prod_{i=1}^n h_i g h_i^{-1} = 1$. A group with generalized torsion element cannot be bi-orderable.

Proof. of Theorem 1 Assume by contradiction there exists a total ordering $<$ of $G(X, S)$ invariant under left and right multiplication. Let $\theta_1, \dots, \theta_n$ denote the n frozen elements in $G(X, S)$. From lemma 3.1, there exists $x_k \in X$ and a frozen element θ_i such that x_k and θ_i do not commute and furthermore, $x_k\theta_i x_k^{-1} = \theta_{f_k^{-1}(i)}$. With no loss of generality, suppose $\theta_i < \theta_{f_k^{-1}(i)}$. So, from our assumption, $x_k\theta_i x_k^{-1} < x_k\theta_{f_k^{-1}(i)}x_k^{-1}$, that is $\theta_{f_k^{-1}(i)} < \theta_{f_k^{-2}(i)}$ from lemma 3.1(i). Inductively, $\theta_i < \theta_{f_k^{-1}(i)} < \theta_{f_k^{-2}(i)} < \dots$. But f_k has finite order, so there exists a natural number p such that $\theta_{f_k^{-p}(i)} = \theta_i$ and this contradicts our assumption. The commutator $[x_k, \theta_i]$ is a generalized torsion element, since on one hand $[x_k, \theta_i] \neq 1$ and $[x_k^p, \theta_i] = 1$ and on second hand $[x_k^p, \theta_i]$ is the product of conjugates of $[x_k, \theta_i]$, using $[x^m, y] = x[x^{m-1}, y]x^{-1}[x, y]$. \square

4. A FAMILY OF STRUCTURE GROUPS WITH A RECURRENT LEFT ORDER

Let $G(X, S)$ be the structure group of a non-degenerate, symmetric set-theoretical solution (X, S) , $|X| = n$. Let $\epsilon : G(X, S) \rightarrow \mathbb{Z}$ denote the augmentation map, defined by $\epsilon(x_{i_1}^{m_1} x_{i_2}^{m_2} \dots x_{i_k}^{m_k}) = \sum_{i=1}^{i=k} m_i$. The map ϵ is an epimorphism (from the form of the defining relations), so $G(X, S)$ is indicable and there is an exact sequence $1 \rightarrow K \rightarrow G(X, S) \rightarrow \mathbb{Z} \rightarrow 1$, with $K = \ker(\epsilon)$. Assume (X, S) satisfies additionally: all the permutations f_1, \dots, f_n are equal. We show that for such solutions the restriction of the bijective 1-cocycle $\pi : G(X, S) \rightarrow \mathbb{Z}^n$ to the normal subgroup K is an isomorphism of groups. Note that for each n , such solutions exist and if all the f_i , $1 \leq i \leq n$, are equal to a cycle of length n , the solution is called a *permutation solution* [19].

Proposition 4.1. *Let $G(X, S)$ be the structure group of a non-degenerate, symmetric (non trivial) set-theoretical solution (X, S) , $|X| = n$, such that all the permutations f_1, \dots, f_n are*

equal. Let K denote the kernel of $\epsilon : G(X, S) \rightarrow \mathbb{Z}$. Let $LO(G(X, S))$ denote the space of left orders of $G(X, S)$. Assume $n \geq 3$. Then

- (i) The restriction of π to K is an isomorphism of groups and $K \simeq \mathbb{Z}^{n-1}$.
- (ii) The normal subgroup K is convex with respect to an infinite number of left orders.
- (iii) There exists a recurrent left order in $LO(G(X, S))$.
- (iv) $LO(G(X, S))$ is homeomorphic to the Cantor set with no isolated element.
- (v) Every left order of $G(X, S)$ is Conradian.

Proof. (i) Let $a_1, a_2 \in K$. From Thm. 2.3, $\pi(a_1 a_2) = (a_2^{-1} \bullet \pi(a_1)) + \pi(a_2)$. As $\epsilon(a_2^{-1}) = \epsilon(a_2) = 0$ and f_1, \dots, f_n are equal to some f , a_2 acts trivially on \mathbb{Z}^n , so $\pi(a_1 a_2) = \pi(a_1) + \pi(a_2)$. As, π is also bijective, it is an isomorphism of groups and clearly $K \simeq \mathbb{Z}^{n-1}$.

(ii) Since $1 \rightarrow K \rightarrow G(X, S) \xrightarrow{\epsilon} \mathbb{Z} \rightarrow 1$ and K, \mathbb{Z} are (left-)orderable, $G(X, S)$ is left-orderable [26][p.26]. Furthermore, each (left-)order of K induces a left order of $G(X, S)$ in the following way: given $g, h \in G(X, S)$, $g \prec h$ if $\epsilon(g) \prec_{\mathbb{Z}} \epsilon(h)$ and if $\epsilon(g) = \epsilon(h)$, $0 \prec_K g^{-1}h$. As there is an uncountable number of orders in K [26][p.43] and each order of K induces a left order of $G(X, S)$, there is an infinite number of left orders of $G(X, S)$ and from the definition of \prec , K is convex (indeed, if $x \prec y \prec z$ and $x, z \in K$, then $\epsilon(y) = 0$).

(iii) $G(X, S)$ is solvable [19] and left-orderable, so it has a recurrent left order [32].

(iv) The space of left orders of a countable (virtually) solvable group is either finite or homeomorphic to the Cantor set [35], so $LO(G(X, S))$ is homeomorphic to the Cantor set. Furthermore, for a countable group G , the space $LO(G)$ is homeomorphic to the Cantor set if and only if it is nonempty and no left order is isolated [17][p.267], so no left order in $LO(G(X, S))$ is isolated. Note, for $n \geq 2$, \mathbb{Z}^n admits no isolated orders [37].

(v) Every left order of $G(X, S)$ is also right invariant on a subgroup of finite index (the free abelian subgroup of finite index N from Section 3), so it is Conradian [17][p.288]. \square

Remark 4.2. In case $n = 2$, there are exactly two solutions: the trivial one with structure group \mathbb{Z}^2 and the permutation solution with structure group the Klein-bottle group (presented by $\langle a, b \mid a^2 = b^2 \rangle$). The construction of left orders from Proposition 4.1 works also in this case, but as $K = \mathbb{Z}$, there are only four left orders induced, which are Conradian. These are all the only left orders and they are isolated (see [18][p.54] for details). The question arises whether, for $n \geq 3$, the construction of left orders from Proposition 4.1 provides also all the left orders.

Let (X, S) be a non-degenerate, symmetric solution with structure group $G(X, S)$. A *retract relation* \equiv on X is a congruence relation defined by $x_i \equiv x_j$ if and only if $f_i = f_j$. The quotient group $G(X, S)/\equiv$ is denoted by $\text{Ret}^1(G)$ and it is also the structure group of a non-degenerate, symmetric solution with set X/\equiv and function S/\equiv induced accordingly. The kernel of the canonical homomorphism $G(X, S) \rightarrow G(X, S)/\equiv$ is a finitely generated torsion-free abelian group [24][p.222]. For any integer $m \geq 1$, $\text{Ret}^{m+1}(G) = \text{Ret}^1(\text{Ret}^m(G))$. The solution (X, S) is called *retractable* if there exists $m \geq 1$ such that $\text{Ret}^m(G)$ is a cyclic group and if m is the smallest such integer, (X, S) is called *retractable (or multipermutation solution) of level m* . A solution (X, S) , $|X| = n$, for which all the permutations f_1, \dots, f_n are equal is retractable of level 1. We refer to [19], [24] for details.

Proof. of Theorem 2 (i), (ii), (iii) Assume (X, S) is retractable of level m . The proof is by induction on m . The case $m = 1$ is proved in Prop. 4.1. For $m \geq 2$, we have the exact sequence $1 \rightarrow \text{Ker}(\equiv) \hookrightarrow G(X, S) \rightarrow G(X, S)/\equiv \rightarrow 1$, where $G(X, S)/\equiv$ is a retractable solution of level $m - 1$. Since $\text{Ker}(\equiv)$ is a torsion free abelian subgroup of $G(X, S)$, it is a (left-)orderable group [28] and from the induction assumption, $G(X, S)/\equiv$ is left-orderable, so $G(X, S)$ is also left-orderable [26][p.26]. Since each left order in $G(X, S)$ is induced by a Conradian left order in $G(X, S)/\equiv$ and all the orders in $\text{Ker}(\equiv)$ are Conradian, $G(X, S)$ has

also an infinite set of Conradian left orders. As $G(X, S)$ is solvable [19], it has a recurrent left order [32] and $LO(G(X, S))$ is homeomorphic to the Cantor set [35]. \square

Some remarks to conclude. There are the following implications: Bi-orderable \Rightarrow Recurrent left-orderable \Rightarrow Locally indicable \Rightarrow Left-orderable \Rightarrow Diffuse \Rightarrow Unique product \Rightarrow Torsion-free and Unique product \Rightarrow Kaplansky's Unit conjecture satisfied (the units in the group algebra are trivial) \Rightarrow Kaplansky's Zero-divisor conjecture satisfied (there are no zero divisors in the group algebra) \Rightarrow Kaplansky's Idempotent conjecture satisfied (there are no non-trivial idempotents in the group algebra). We found there are no (non-trivial) solutions with structure group bi-orderable, and all the retractable solutions admit a recurrent left order. For non-retractable solutions, the kernel of $\epsilon : G(X, S) \rightarrow \mathbb{Z}$ is not necessarily a free abelian group and the methods from Proposition 4.1 cannot be applied, so, the question arises whether there exist structure groups of non-retractable solutions that are left-orderable. In fact, there exist non-retractable solutions with structure group a non unique product group. Indeed, E. Jespers and J. Okninski give an example of structure group of a non-retractable solution (with $n = 4$) which is not a unique product group [24][p.224], as they prove this group contains a subgroup isomorphic to the Promislow group. The Promislow group is a non unique product Bieberbach group [33], which does not belong to the class of the structure groups [24][p.224]. So, this answers in the negative the question whether every Garside group is left-orderable and furthermore provides an example of non unique product Garside group. So, being a Garside group does not imply being locally indicable, nor left-orderable, nor unique product, but a Garside group is necessarily torsion free [12].

The structure group of a non-degenerate, symmetric solution enjoys another interesting particularity, it is a Bieberbach group [20], [24], that is it is a torsion free crystallographic group. A classification of Bieberbach groups of small dimension (up to four) in relation with the existence of a left order is given in [25][p.13]. Bieberbach groups satisfy Kaplansky's zero divisor conjecture, as it holds for all torsion-free finite-by-solvable groups [27]. As the braid groups are left-orderable, they also satisfy the zero divisor conjecture, so Question 3.3 from [17] might be replaced by: does a Garside group satisfy Kaplansky's zero divisor conjecture? It is still unknown whether Kaplansky's unit conjecture holds in Bieberbach groups. D. Craven and P. Pappas study the question whether the unit conjecture holds in the Promislow group (also called Passman group) (see [11] for some preliminary results).

Amongst the 23 solutions with $n = 4$, there are two non-retractable (indecomposable) solutions and the structure group of one of them is presented by $\text{Gp}\langle x_1, x_2, x_3, x_4 \mid x_1^2 = x_2^2, x_1x_2 = x_3^2, x_2x_1 = x_4^2, x_1x_3 = x_4x_1, x_2x_4 = x_3x_2, x_3x_4 = x_4x_3 \rangle$ with $g_1 = (1, 2, 3, 4)$, $g_2 = (1, 4, 3, 2)$, $g_3 = (1, 3)$, $g_4 = (2, 4)$; $f_1 = (1, 2, 4, 3)$, $f_2 = (1, 3, 4, 2)$, $f_3 = (2, 3)$, $f_4 = (1, 4)$. The second one is given in [24][p.224]. If a decomposable solution contains a non-retractable solution with non unique product structure group, then it is also non-retractable and it has non unique product structure group. So, the question is what about the class of indecomposable non-retractable solutions? For $5 \leq n \leq 7$, all the indecomposable solutions are retractable and for $n = 8$, amongst the 34528 solutions, there are 47 indecomposable non-retractable solutions [19]. As the structure groups have a wide range of behaviours, the intriguing question is whether amongst the indecomposable non-retractable solutions, there are special cases of groups. More specifically, can we find there groups that are left-orderable, or diffuse? Or, groups that are unique product but not left-orderable? Or, diffuse but not left-orderable?

REFERENCES

- [1] R. Botto Mura and A.H. Rhemtulla, *Orderable groups*, Lecture Notes in Pure and Applied Mathematics **27**, Marcel Dekker Inc., New York 1977.
- [2] B.H. Bowditch, *A variation on the unique product property*, J. London Math. Soc. (2) **62** (2000), no. 3, 813-826.

- [3] S.D. Brodski, *Equations over groups, and groups with one defining relation*, Sibirsk. Mat. Zh. **25** (1984), 84-103.
- [4] L.S. Charlap, *Bieberbach groups and flat manifolds*, Springer-Verlag, 1986.
- [5] I.M. Chiswell, P. Kropholler, *Soluble right orderable groups are locally indicable*, Canad. Math. Bull. **36** (1993), 22-29.
- [6] F. Chouraqui, *Garside groups and the Yang-Baxter equation*, Comm. in Algebra **38** (2010) 4441-4460.
- [7] F. Chouraqui and E. Godelle, *Folding of set-theoretical solutions of the Yang-Baxter Equation*, Algebra and Representation Theory **15** (2012) 1277-1290.
- [8] F. Chouraqui and E. Godelle, *Finite quotients of I-type groups*, Adv. in Math. **258** (2014) 46-68.
- [9] F. Chouraqui, *Construction of a group of automorphisms for an infinite family of Garside groups*, Preprint, Arxiv 1411.1189.
- [10] P.F. Conrad, *Right-ordered groups*, Michigan Math. J. **6** (1959), 267-275.
- [11] D.A. Craven, P. Pappas, *On the unit conjecture for supersoluble group algebras*, J. Algebra **394** (2013), 310-356.
- [12] P. Dehornoy, *Gaussian groups are torsion free*, J. of Algebra **210** (1998) 291-297.
- [13] P. Dehornoy and L. Paris, *Gaussian groups and Garside groups, two generalisations of Artin groups*, Proc. London Math Soc. **79** (1999) 569-604.
- [14] P. Dehornoy, *Coxeter-like groups for groups of set-theoretic solutions of the Yang-Baxter equation*, Comptes Rendus Mathematiques **351** (2013) 419-424.
- [15] P. Dehornoy, F. Digne, E. Godelle, D. Krammer and J. Michel, *Foundations of Garside Theory*, Europ. Math. Soc. Tracts in Mathematics, <http://www.math.unicaen/~garside>.
- [16] P. Dehornoy, *Braid groups and left-distributive operations*, Trans. Amer. Math. Soc. **345** (1994), no. 1, 115-150.
- [17] P. Dehornoy, I. Dynnikov, D. Rolfsen and B. Wiest, *Ordering braids*, Mathematical Surveys and Monographs 148, American Mathematical Society, Providence, RI, 2008.
- [18] B. Deroin, A. Navas, C. Rivas, *Groups, Orders, and Dynamics*, Preprint, ArXiv 1408.5805.
- [19] P. Etingof, T. Schedler, A. Soloviev, *Set-theoretical solutions to the Quantum Yang-Baxter equation*, Duke Math. J. **100** (1999) 169-209.
- [20] T. Gateva-Ivanova and M. Van den Bergh, *Semigroups of I-type*, J. Algebra **206** (1998) 97-112.
- [21] E. Ghys, *Groups acting on the circle*, L'Enseignement Mathematique **47** (2001), 239-407.
- [22] A.M.W. Glass, *Partially ordered groups*, Series in Algebra **7**, World Scientific, River Edge, NJ (1999).
- [23] E. Jespers, J. Okninski, *Monoids and groups of I-type*, Algebra and Representation Theory **8** (2005) 709-729.
- [24] E. Jespers, J. Okninski, *Noetherian Semigroup Algebras*, Algebra and applications, vol.7 (2007).
- [25] S. Kionke, J. Raimbault, *On Geometric aspects of diffuse groups*, Preprint, Arxiv 1441.6449.
- [26] V. Kopytov, N. Ya Medvedev, *Right-ordered groups*, Siberian school of Algebra and Logic, Consultants bureau, New York 1996.
- [27] P.H. Kropholler, P.A. Linnell, and J.A. Moody, *Applications of a new K-theoretic theorem to soluble group rings*, Proc. Amer. Math. Soc. **104** (1988), no. 3, 675-684.
- [28] F.W. Levi, *Ordered groups*, Proc. Indian Acad. Sci. A. **16** (1942), 256-263.
- [29] P.A. Linnell, *The space of left orders of a group is either finite or uncountable*, Bull. London Math. Soc. **43** (2011), no. 1, 200-202.
- [30] P.A. Linnell, *Left ordered amenable and locally indicable groups*, J. London Math. Soc. **60** (1999), 133-142.
- [31] P.A. Linnell, D. Morris-Witte, *Amenable groups with a locally invariant order are locally indicable*, Groups, Geometry and dynamics **8** (2014), n.2, 467-478.
- [32] D. Morris-Witte, *Amenable groups that act on the line*, Algebr. Geom. Topol. **6** (2006), 2509-2518.
- [33] S.D. Promislow, *A simple example of a torsion free, nonunique product group*, Bull. London Math. Soc. **20** (1998), 302-304.
- [34] A.H. Rhemtulla, D. Rolfsen, *Local indicability in ordered groups: braids and elementary amenable groups*, Proc. Amer. Math. Soc. **130** (2002), n.9, 2569-2577.
- [35] C. Rivas, R. Tessera, *On the space of left-orderings of virtually solvable groups*, Preprint, Arxiv 1209.3251.
- [36] D. Rolfsen, *Low dimensional topology and ordering groups*, Math. Slovaca. **64**, n.3 (2014), 579-600.
- [37] A.S. Sikora, *Topology on the spaces of orderings of groups*, Bull. London Math. Soc. **36** (2004), n.4, 519-526.

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